

# PRIMES is in P

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## Abstract

We present an unconditional deterministic polynomial-time algorithm that determines whether an input number is prime or composite.

## 1 Introduction

Prime numbers are of fundamental importance in mathematics in general, and number theory in particular. So it is of great interest to study different properties of prime numbers. Of special interest are those properties that allow one to efficiently determine if a number is prime. Such efficient tests are also useful in practice: a number of cryptographic protocols need large prime numbers.

Let PRIMES denote the set of all prime numbers. The definition of prime numbers already gives a way of determining if a number  $n$  is in PRIMES: try dividing  $n$  by every number  $m \leq \sqrt{n}$ —if any  $m$  divides  $n$  then it is composite otherwise prime. This test was already known since the time of ancient Greeks—it is a specialization of the *Sieve of Eratosthenes* (ca. 240 BC) that generates all primes less than  $n$ . The test, however, is inefficient: it takes  $\Omega(\sqrt{n})$  steps to determine if  $n$  is prime. An efficient test should need only a polynomial (in the size of the input =  $\lceil \log n \rceil$ ) number of steps. A property that *almost* gives an efficient test is Fermat’s Little Theorem: for any prime number  $p$ , and any number  $a$  not divisible by  $p$ ,  $a^{p-1} = 1 \pmod{p}$ . Given an  $a$  and  $n$  it can be efficiently checked if  $a^{n-1} = 1 \pmod{n}$  using repeated squaring to compute  $(n-1)^{th}$  power of  $a$ . However, it is not a correct test since many composites  $n$  also satisfy it for some  $a$ ’s (*all*  $a$ ’s in case of *Carmichael* numbers [Car10]). Nevertheless, Fermat’s Little Theorem became the basis for many efficient primality tests.

Since the beginning of complexity theory in 1960s—when the notions of problem complexity were formalized and various complexity classes were defined—this problem (referred as *primality testing* problem) has been investigated intensively. It is trivial to see that the problem is in the class co-NP: if  $n$  is not prime it has an easily verifiable short certificate, viz., a non-trivial factor of  $n$ . In 1974, Pratt observed that the problem is in the class NP too [Pra75] (thus putting it in  $NP \cap co-NP$ ).

In 1975, Miller [Mil76] used a property based on Fermat’s Little Theorem to obtain a deterministic polynomial-time algorithm for primality testing assuming *Extended Riemann Hypothesis (ERH)*. Within a year, his test was modified by Rabin [Rab80] to yield an unconditional but randomized polynomial-time algorithm. Independently, Solovay and Strassen [SS77] obtained, in 1974, a different randomized polynomial-time algorithm using the property that for a prime  $n$ ,  $\left(\frac{a}{n}\right) = a^{\frac{n-1}{2}} \pmod{n}$  for every  $a$  ( $(-)$  is the Jacobi symbol). Their algorithm can also be made deterministic under ERH. Since then, a number of randomized polynomial-time algorithms have been proposed for primality testing based on many different properties.

In 1983, Adleman, Pomerance, and Rumely achieved a major breakthrough by giving a deterministic algorithm for primality that runs in  $(\log n)^{O(\log \log \log n)}$  time (all the previous deterministic algorithms

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required exponential time). Their algorithm was (in a sense) a generalization of Miller’s idea and used higher reciprocity laws. In 1986, Goldwasser and Kilian [GK86] proposed a randomized algorithm based on Elliptic curves running in expected polynomial-time on almost all inputs (*all* inputs under a widely believed hypothesis) that produces an easily verifiable short certificate for primality (until then, all randomized algorithms produced certificates for compositeness only). Based on their ideas, a similar algorithm was developed by Atkin [Atk86]. Adleman and Huang [AH92] modified Goldwasser-Kilian algorithm to obtain a randomized algorithm that runs in expected polynomial-time on all inputs.

The ultimate goal of this line of research has been, of course, to obtain an unconditional deterministic polynomial-time algorithm for primality testing. Despite the impressive progress made so far, this goal has remained elusive. In this paper, we achieve this. We give a deterministic,  $O^{\sim}(\log^{7.5} n)$  time algorithm for testing if a number is prime. Heuristically, our algorithm does better: under a widely believed conjecture on the density of Sophie Germain primes (primes  $p$  such that  $2p+1$  is also prime), the algorithm takes only  $O^{\sim}(\log^6 n)$  steps. Our algorithm is based on a generalization of Fermat’s Little Theorem to polynomial rings over finite fields. Notably, the correctness proof of our algorithm requires only simple tools of algebra (except for appealing to a sieve theory result on the density of primes  $p$  with  $p-1$  having a large prime factor—and even this is not needed for proving a weaker time bound of  $O^{\sim}(\log^{10.5} n)$  for the algorithm). In contrast, the correctness proofs of deterministic algorithms of [APR83, GK86, Atk86] are much more complex.

In section 2, we summarize the basic idea behind our algorithm. In section 3, we fix the notation used. In section 4, we state the algorithm and present its proof of correctness. In section 5, we obtain bounds on the running time of the algorithm. Section 6 discusses some ways of improving the time complexity of the algorithm.

## 2 The Idea

Our test is based on the following identity for prime numbers which is a generalization of Fermat’s Little Theorem. This identity was the basis for a randomized polynomial-time algorithm in [AB99]:

**Lemma 2.1.** *Let  $a \in \mathcal{Z}$ ,  $n \in \mathcal{N}$ ,  $n \geq 2$ , and  $(a, n) = 1$ . Then  $n$  is prime if and only if*

$$(X + a)^n = X^n + a \pmod{n}. \tag{1}$$

*Proof.* For  $0 < i < n$ , the coefficient of  $x^i$  in  $((X + a)^n - (X^n + a))$  is  $\binom{n}{i} a^{n-i}$ .

Suppose  $n$  is prime. Then  $\binom{n}{i} = 0 \pmod{n}$  and hence all the coefficients are zero.

Suppose  $n$  is composite. Consider a prime  $q$  that is a factor of  $n$  and let  $q^k || n$ . Then  $q^k$  does not divide  $\binom{n}{q}$  and is coprime to  $a^{n-q}$  and hence the coefficient of  $X^q$  is not zero  $\pmod{n}$ . Thus  $((X + a)^n - (X^n + a))$  is not identically zero over  $Z_n$ .  $\square$

The above identity suggests a simple test for primality: given input  $n$ , choose an  $a$  and test whether the congruence (1) is satisfied. However, this takes time  $\Omega(n)$  because we need to evaluate  $n$  coefficients in the LHS in the worst case. A simple way to reduce the number of coefficients is to evaluate both sides of (1) modulo a polynomial of the form  $X^r - 1$  for an appropriately chosen small  $r$ . In other words, test if the following equation is satisfied:

$$(X + a)^n = X^n + a \pmod{X^r - 1, n}. \tag{2}$$

From Lemma 2.1 it is immediate that all primes  $n$  satisfy the equation (2) for all values of  $a$  and  $r$ . The problem now is that some composites  $n$  may also satisfy the equation for a few values of  $a$  and  $r$  (and indeed they do). However, we can almost restore the characterization: we show that for appropriately chosen  $r$  if the equation (2) is satisfied for several  $a$ ’s then  $n$  must be a prime power. The number of  $a$ ’s and the appropriate  $r$  are both bounded by a polynomial in  $\log n$  and therefore, we get a deterministic polynomial time algorithm for testing primality.

## 3 Notation and Preliminaries

The class P is the class of sets accepted by deterministic polynomial-time Turing machines [Lee90]. See [Lee90] for the definitions of classes NP, co-NP, etc.

$F_p$  denotes the finite field with  $p$  elements, where  $p$  is prime. Recall that if  $p$  is prime and  $h(X)$  is a polynomial of degree  $d$  and irreducible in  $F_p$ , then  $F_p[X]/(h(X))$  is a finite field of order  $p^d$ . We will use the notation  $f(X) = g(X) \pmod{h(X), n}$  to represent equation  $f(X) = g(X)$  in the ring  $Z_n[X]/(h(X))$ .

We use the symbol  $O^\sim(t(n))$  for  $O(t(n) \cdot \text{poly}(\log t(n)))$ , where  $t(n)$  is any function of  $n$ . For example,  $O^\sim(\log^k n) = O(\log^k n \cdot \text{poly}(\log \log n)) = O(\log^{k+\epsilon} n)$  for any  $\epsilon > 0$ . Unless stated otherwise, log will be base 2 in this paper.

$\mathcal{N}$  and  $\mathcal{Z}$  denote the set of natural numbers and integers respectively. Given  $r \in \mathcal{N}$ ,  $a \in \mathcal{Z}$  with  $(a, r) = 1$ , the *order of  $a$  modulo  $r$*  is the smallest number  $k$  such that  $a^k = 1 \pmod{r}$ . It is denoted as  $o_r(a)$ . For  $r \in \mathcal{N}$ ,  $\phi(r)$  is *Euler's totient function* giving the number of numbers less than  $r$  that are relatively prime to  $r$ . It is easy to see that  $o_r(a) \mid \phi(r)$  for any  $a$ ,  $(a, r) = 1$ .

We will need the following simple fact about the lcm of first  $m$  numbers (see, e.g., [Nai82] for a proof).

**Lemma 3.1.** *Let  $LCM(m)$  denote the lcm of first  $m$  numbers. For  $m \geq 7$ :*

$$LCM(m) \geq 2^m.$$

## 4 The Algorithm and Its Correctness

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Input: integer  $n > 1$ .
1. If  $(n = a^b$  for  $a \in \mathcal{N}$  and  $b > 1)$ , output COMPOSITE.
2. Find the smallest  $r$  such that  $o_r(n) > 4 \log^2 n$ .
3. If  $1 < (a, n) < n$  for some  $a \leq r$ , output COMPOSITE.
4. If  $n \leq r$ , output PRIME.
5. For  $a = 1$  to  $\lfloor 2\sqrt{\phi(r)} \log n \rfloor$  do
       if  $((X + a)^n \neq X^n + a \pmod{X^r - 1, n})$ , output COMPOSITE;
6. Output PRIME;

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### Algorithm for Primality Testing

**Theorem 4.1.** *The algorithm above returns PRIME if and only if  $n$  is prime.*

In the remainder of the section, we establish this theorem through a sequence of lemmas. The following is trivial:

**Lemma 4.2.** *If  $n$  is prime, the algorithm returns PRIME.*

*Proof.* If  $n$  is prime then steps 1 and 3 can never return COMPOSITE. By Lemma 2.1, the for loop also cannot return COMPOSITE. Therefore the algorithm will identify  $n$  as PRIME either in step 4 or in step 6.  $\square$

The converse of the above lemma requires a little more work. If the algorithm returns PRIME in step 4 then  $n$  must be prime since otherwise step 3 would have found a non-trivial factor of  $n$ . So the only remaining case is when the algorithm returns PRIME in step 6. For the purpose of subsequent analysis we assume this to be the case.

The algorithm has two main steps (2 and 5): step 2 finds an appropriate  $r$  and step 5 verifies the equation (2) for a number of  $a$ 's. We first bound the magnitude of the appropriate  $r$ .

**Lemma 4.3.** *There exist an  $r \leq \lceil 16 \log^5 n \rceil$  such that  $o_r(n) > 4 \log^2 n$ .*

*Proof.* Let  $r_1, r_2, \dots, r_t$  be all numbers such that  $o_{r_i}(n) \leq 4 \log^2 n$ . Each of these numbers must divide the product

$$\prod_{i=1}^{\lfloor 4 \log^2 n \rfloor} (n^i - 1) < n^{16 \log^4 n} \leq 2^{16 \log^5 n}.$$

By Lemma 3.1, the lcm of first  $\lceil 16 \log^5 n \rceil$  numbers is at least  $2^{\lceil 16 \log^5 n \rceil}$  and therefore there must exist a number  $r \leq \lceil 16 \log^5 n \rceil$  such that  $o_r(n) > 4 \log^2 n$ .  $\square$

Let  $p$  be a prime divisor of  $n$ . We have  $p > r$  since otherwise either step 3 or step 4 would decide about primality of  $n$ . Since  $(n, r) = 1$  (otherwise either step 3 or step 4 will correctly identify  $n$ ),  $p, n \in Z_r^*$ . Numbers  $p$  and  $r$  will be fixed in the remainder of this section. Also, let  $\ell = \lfloor 2\sqrt{\phi(r)} \log n \rfloor$ .

Step 5 of the algorithm verifies  $\ell$  equations. Since the algorithm does not output COMPOSITE in this step, we have:

$$(X + a)^n = X^n + a \pmod{X^r - 1, n}$$

for every  $a$ ,  $1 \leq a \leq \ell$ . This implies:

$$(X + a)^n = X^n + a \pmod{X^r - 1, p} \quad (3)$$

for  $1 \leq a \leq \ell$ . By Lemma 2.1, we have:

$$(X + a)^p = X^p + a \pmod{X^r - 1, p} \quad (4)$$

for  $1 \leq a \leq \ell$ . Thus  $n$  behaves like prime  $p$  in the above equation. We give a name to this property:

**Definition 4.4.** For polynomial  $f(X)$  and number  $m \in \mathcal{N}$ , we say that  $m$  is *introspective* for  $f(X)$  if

$$[f(X)]^m = f(X^m) \pmod{X^r - 1, p}.$$

It is clear from equations (3) and (4) that both  $n$  and  $p$  are introspective for  $X + a$  for  $1 \leq a \leq \ell$ . The following lemma shows that introspective numbers are closed under multiplication:

**Lemma 4.5.** *If  $m$  and  $m'$  are introspective numbers for  $f(X)$  then so is  $m \cdot m'$ .*

*Proof.* Since  $m$  is introspective for  $f(X)$  we have:

$$[f(X)]^{m \cdot m'} = [f(X^m)]^{m'} \pmod{X^r - 1, p}.$$

Also, since  $m'$  is introspective for  $f(X)$ , we have (after replacing  $X$  by  $X^m$  in the equation):

$$\begin{aligned} [f(X^m)]^{m'} &= f(X^{m \cdot m'}) \pmod{X^{m \cdot r} - 1, p} \\ &= f(X^{m \cdot m'}) \pmod{X^r - 1, p} \text{ (since } X^r - 1 \text{ divides } X^{m \cdot r} - 1 \text{)}. \end{aligned}$$

Putting together the above two equations we get:

$$[f(X)]^{m \cdot m'} = f(X^{m \cdot m'}) \pmod{X^r - 1, p}.$$

$\square$

For number  $m$ , the set of polynomials for which it is introspective is also closed under multiplication:

**Lemma 4.6.** *If  $m$  is introspective for  $f(X)$  and  $g(X)$  then it is also introspective for  $f(X) \cdot g(X)$ .*

*Proof.* We have:

$$\begin{aligned} [f(X) \cdot g(X)]^m &= [f(X)]^m \cdot [g(X)]^m \\ &= f(X^m) \cdot g(X^m) \pmod{X^r - 1, p}. \end{aligned}$$

$\square$

The above two lemmas together imply that every number in the set  $I = \{n^i \cdot p^j \mid i, j \geq 0\}$  is introspective for every polynomials in the set  $P = \{\prod_{a=1}^{\ell} (X + a)^{e_a} \mid e_a \geq 0\}$ . We now define two groups based on these sets that will play a crucial role in the proof.

The first group is the set of all residues of numbers in  $I$  modulo  $r$ . This is a subgroup of  $Z_r^*$  since, as already observed,  $(n, r) = (p, r) = 1$ . Let  $G$  be this group and  $|G| = t$ .  $G$  is generated by  $n$  and  $p$  modulo  $r$  and since  $o_r(n) > 4 \log^2 n$ ,  $t > 4 \log^2 n$ .

To define the second group, we need some basic facts about cyclotomic polynomials over finite fields. Let  $Q_r(X)$  be  $r^{\text{th}}$  cyclotomic polynomial over  $F_p$ . Polynomial  $Q_r(X)$  divides  $X^r - 1$  and factors into irreducible factors of degree  $o_r(p)$  [LN86]. Let  $h(X)$  be one such irreducible factor. The second group is the set of all non-zero residues of polynomials in  $P$  modulo  $h(X)$  and  $p$ . Let  $\mathcal{G}$  be this group. This group is generated by  $X + 1, X + 2, \dots, X + \ell$  in the field  $F = F_p[X]/(h(X))$  and is a subgroup of the multiplicative group of  $F_p[X]/(h(X))$ .

The size of group  $\mathcal{G}$  is exponential in  $t$ :

**Lemma 4.7.**  $|\mathcal{G}| \geq \binom{t+\ell-2}{t-1}$ .<sup>1</sup>

*Proof.* First note that since  $h(X)$  is a factor of the cyclotomic polynomial  $Q_r(X)$ ,  $X$  is a primitive  $r^{\text{th}}$  root of unity in  $F$ .

We now show that any two distinct polynomials of degree less than  $t$  in  $P$  will map to different elements in  $\mathcal{G}$ . Let  $f(X)$  and  $g(X)$  be two such polynomials in  $P$ . Suppose  $f(X) = g(X)$  in the field  $F$ . Let  $m \in I$ . We also have  $[f(X)]^m = [g(X)]^m$  in  $F$ . Since  $m$  is introspective for both  $f$  and  $g$ , and  $h(X)$  divides  $X^r - 1$ , we get:

$$f(X^m) = g(X^m)$$

in  $F$ . This implies that  $X^m$  is a root of the polynomial  $Q(Y) = f(Y) - g(Y)$  for every  $m \in G$ . Since  $(m, r) = 1$  ( $G$  is a subgroup of  $Z_r^*$ ), each such  $X^m$  is a primitive  $r^{\text{th}}$  root of unity. Hence there will be  $|G| = t$  distinct roots of  $Q(Y)$  in  $F$ . However, the degree of  $Q(Y)$  is less than  $t$  by the choice of  $f$  and  $g$ . This is a contradiction and therefore,  $f(X) \neq g(X)$  in  $F$ .

Note that  $i \neq j$  in  $F_p$  for  $1 \leq i \neq j \leq \ell$  since  $\ell = \lfloor 2\sqrt{\phi(r)} \log n \rfloor < 2\sqrt{r} \log n < r$  and  $p > r$ . So the elements  $X + 1, X + 2, \dots, X + \ell$  are all distinct in  $F$ . It is possible that  $X + a = 0$  in  $F$  for some  $a \leq \ell$  (this happens when  $h(X) = X + a$ ), and this  $X + a$  will not be included in the group  $\mathcal{G}$ . So there exist at least  $\ell - 1$  distinct polynomials of degree one in  $\mathcal{G}$ . Therefore, there exist at least  $\binom{t+\ell-2}{t-1}$  distinct polynomials of degree  $< t$  in  $\mathcal{G}$ .  $\square$

In case  $n$  is not a power of  $p$ , size of  $\mathcal{G}$  is also upper bounded by an exponential function in  $t$ :

**Lemma 4.8.** *If  $n$  is not a power of  $p$  then  $|\mathcal{G}| < \frac{1}{2}n^{2\sqrt{t}}$ .*

*Proof.* Consider the following subset of  $I$ :

$$\hat{I} = \{n^i \cdot p^j \mid 0 \leq i, j \leq \lfloor \sqrt{t} \rfloor\}.$$

If  $n$  is not a power of  $p$  then the set  $\hat{I}$  has  $(\lfloor \sqrt{t} \rfloor + 1)^2 > t$  distinct numbers. Since  $|G| = t$ , at least two numbers in  $\hat{I}$  must be equal modulo  $r$ . Let these be  $m_1$  and  $m_2$  with  $m_1 > m_2$ . So we have:

$$X^{m_1} = X^{m_2} \pmod{X^r - 1}.$$

Let  $f(X) \in P$ . Then,

$$\begin{aligned} [f(X)]^{m_1} &= f(X^{m_1}) \pmod{X^r - 1, p} \\ &= f(X^{m_2}) \pmod{X^r - 1, p} \\ &= [f(X)]^{m_2} \pmod{X^r - 1, p}. \end{aligned}$$

This implies

$$[f(X)]^{m_1} = [f(X)]^{m_2}$$

in the field  $F$ . Therefore,  $f(X) \in \mathcal{G}$  is a root of the polynomial  $Q'(Y) = Y^{m_1} - Y^{m_2}$  in the field  $F$ .<sup>2</sup> As  $f(X)$  is an arbitrary element of  $\mathcal{G}$ , we have that the polynomial  $Q'(Y)$  has at least  $|G|$  distinct roots in  $F$ . The degree of  $Q'(Y)$  is  $m_1 \leq (np)^{\lfloor \sqrt{t} \rfloor} < \frac{1}{2}n^{2\sqrt{t}}$  (since  $p$  divides  $n$  but  $n \neq p$  by assumption). This shows  $|\mathcal{G}| < \frac{1}{2}n^{2\sqrt{t}}$ .  $\square$

Armed with these estimates on the size of  $\mathcal{G}$ , we are now ready to prove the correctness of the algorithm:

<sup>1</sup>This lemma improves a bound shown in an earlier version of our paper [AKS02]. It was first observed by Hendrik Lenstra Jr. in a slightly weaker form [Len02].

<sup>2</sup>This formulation of the argument is by Adam Kalai, Amit Sahai, and Madhu Sudan [KSS02].

**Lemma 4.9.** *If the algorithm returns PRIME then  $n$  is prime.*

*Proof.* Suppose that the algorithm returns PRIME. Lemma 4.7 implies that for  $t = |G|$  and  $\ell = \lfloor 2\sqrt{\phi(r)} \log n \rfloor$ :

$$\begin{aligned} |G| &\geq \binom{t + \ell - 2}{t - 1} \\ &\geq \binom{\ell - 1 + \lfloor 2\sqrt{t} \log n \rfloor}{\lfloor 2\sqrt{t} \log n \rfloor} \quad (\text{since } t > 2\sqrt{t} \log n) \\ &\geq \binom{2\lfloor 2\sqrt{t} \log n \rfloor - 1}{\lfloor 2\sqrt{t} \log n \rfloor} \quad (\text{since } \ell = \lfloor 2\sqrt{\phi(r)} \log n \rfloor \geq \lfloor 2\sqrt{t} \log n \rfloor) \\ &\geq 2^{\lfloor 2\sqrt{t} \log n \rfloor} \quad (\text{since } 2\sqrt{t} \log n \geq 3) \\ &\geq \frac{1}{2} n^{2\sqrt{t}}. \end{aligned}$$

By Lemma 4.8,  $|G| < \frac{1}{2} n^{2\sqrt{t}}$  if  $n$  is not a power of  $p$ . Therefore,  $n = p^k$  for some  $k > 0$ . If  $k > 1$  then the algorithm will return COMPOSITE in step 1. Therefore,  $n = p$ .  $\square$

This completes the proof of theorem.

## 5 Time Complexity Analysis and Improvements

It is straightforward to calculate the time complexity of the algorithm. In these calculations we use the fact that addition, multiplication, and division operations between two  $m$  bits numbers can be performed in time  $O^\sim(m)$  [vzGG99]. Similarly, these operations on two degree  $d$  polynomials with coefficients at most  $m$  bits in size can be done in time  $O^\sim(d \cdot m)$  steps [vzGG99].

**Theorem 5.1.** *The asymptotic time complexity of the algorithm is  $O^\sim(\log^{10.5} n)$ .*

*Proof.* The first step of the algorithm takes asymptotic time  $O^\sim(\log^3 n)$  [vzGG99].

In step 2, we find an  $r$  with  $o_r(n) > 4 \log^2 n$ . This can be done by trying out successive values of  $r$  and testing if  $n^k \not\equiv 1 \pmod{r}$  for every  $k \leq 4 \log^2 n$ . For a particular  $r$ , this will involve at most  $O(\log^2 n)$  multiplications modulo  $r$  and so will take time  $O^\sim(\log^2 n \log r)$ . By Lemma 4.3 we know that only  $O(\log^5 n)$  different  $r$ 's need to be tried. Thus the total time complexity of step 2 is  $O^\sim(\log^7 n)$ .

The third step involves computing gcd of  $r$  numbers. Each gcd computation takes time  $O(\log n)$  [vzGG99], and therefore, the time complexity of this step is  $O(r \log n) = O(\log^6 n)$ . The time complexity of step 4 is just  $O(\log n)$ .

In step 5, we need to verify  $\lfloor 2\sqrt{\phi(r)} \log n \rfloor$  equations. Each equation requires  $O(\log n)$  multiplications of degree  $r$  polynomials with coefficients of size  $O(\log n)$ . So each equation can be verified in time  $O^\sim(r \log^2 n)$  steps. Thus the time complexity of step 5 is  $O^\sim(r \sqrt{\phi(r)} \log^3 n) = O^\sim(r^{\frac{3}{2}} \log^3 n) = O^\sim(\log^{10.5} n)$ . This time dominates all the other and is therefore the time complexity of the algorithm.  $\square$

The time complexity of the algorithm can be improved by improving the estimate for  $r$  (done in Lemma 4.3). Of course the best possible scenario would be when  $r = O(\log^2 n)$  and in that case the time complexity of the algorithm would be  $O^\sim(\log^6 n)$ . In fact, there are two conjectures that support the possibility of such an  $r$  (below  $\ln$  is natural logarithm):

**Artin's Conjecture:** Given any number  $n \in \mathcal{N}$  that is not a perfect square, the number of primes  $q \leq m$  for which  $o_q(n) = q - 1$  is asymptotically  $A(n) \cdot \frac{m}{\ln m}$  where  $A(n)$  is Artin's constant with  $A(n) > 0.35$ .

**Sophie-Germain Prime Density Conjecture:** The number of primes  $q \leq m$  such that  $2q + 1$  is also a prime is asymptotically  $\frac{2C_2 m}{\ln^2 m}$  where  $C_2$  is the twin prime constant (estimated to be approximately 0.66). Primes  $q$  with this property are called Sophie-Germain primes.

Artin’s conjecture—if it becomes effective for  $m = O(\log^2 n)$ —immediately shows that there is an  $r = O(\log^2 n)$  with the required properties. There has been some progress towards proving Artin’s conjecture [GM84, GMM85, HB86], and it is also known that this conjecture holds under Generalized Riemann Hypothesis.

If the second conjecture holds, we can conclude that  $r = O^\sim(\log^2 n)$ :

By density of Sophie-Germain primes, there must exist at least  $\log^2 n$  such primes between  $8 \log^2 n$  and  $c \log^2 n (\log \log n)^2$  for suitable constant  $c$ . For any such prime  $q$ , either  $o_q(n) \leq 2$  or  $o_q(n) \geq \frac{q-1}{2}$ . Any  $q$  for which  $o_q(n) \leq 2$  must divide  $(n-1)(n^2-1)$  and so the number of such  $q$  is bounded by  $O(\log n)$ . This implies that there must exist a prime  $r = O^\sim(\log^2 n)$  such that  $o_r(n) \geq 4 \log^2 n$ . Such an  $r$  will yield an algorithm with time complexity  $O^\sim(\log^6 n)$ .

There has been progress towards proving this conjecture as well. Let  $P(m)$  denote the greatest prime divisor of number  $m$ . Goldfeld [Gol69] showed that primes  $q$  with  $P(q-1) > q^{\frac{1}{2}+c}$ ,  $c \approx \frac{1}{12}$ , occur with positive density. Improving upon this, Fouvry has shown:

**Lemma 5.2.** [Fou85] *There exist constants  $c > 0$  and  $n_0$  such that, for all  $x \geq n_0$ :*

$$|\{q \mid q \text{ is prime, } q \leq x \text{ and } P(q-1) > q^{\frac{1}{2}+c}\}| \geq c \frac{x}{\ln x}.$$

The above lemma is now known to hold for exponents up to 0.6683 [BH96]. Using the above lemma, we can improve the analysis of our algorithm:

**Theorem 5.3.** *The asymptotic time complexity of the algorithm is  $O^\sim(\log^{7.5} n)$ .*

*Proof.* As argued above, a high density of primes  $q$  with  $P(q-1) > q^{\frac{1}{2}+c}$  implies that step 2 of the algorithm will find an  $r = O(\log^3 n)$  with  $o_r(n) > 4 \log^2 n$ . This brings the complexity of the algorithm down to  $O^\sim(\log^{7.5} n)$ .  $\square$

## 6 Future Work

In our algorithm, the `for` loop in step 5 needs to run for  $\lfloor 2\sqrt{\phi(r)} \log n \rfloor$  times to ensure that the size of the group  $\mathcal{G}$  is large enough. The number of iterations of the loop could be reduced if we can show that a still smaller set of  $(X+a)$ ’s generates a group of the required size. This seems very likely.

One can further improve the complexity to  $O^\sim(\log^3 n)$  if the following conjecture—given in [BP01] and verified for  $r \leq 100$  and  $n \leq 10^{10}$  in [KS02]—is proved:

**Conjecture.** *If  $r$  is a prime number that does not divide  $n$  and if*

$$(X-1)^n = X^n - 1 \pmod{X^r - 1, n}, \tag{5}$$

*then either  $n$  is prime or  $n^2 = 1 \pmod{r}$ .*

If this conjecture is true, we can modify the algorithm slightly to first search for an  $r$  which does not divide  $n^2 - 1$ . Such an  $r$  can assuredly be found in the range  $[2, 4 \log n]$ . This is because the product of prime numbers less than  $x$  is at least  $e^x$  (see [Apo97]). Thereafter we can test whether the congruence (5) holds or not. Verifying the congruence takes time  $O^\sim(r \log^2 n)$ . This gives a time complexity of  $O^\sim(\log^3 n)$ .

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